J. Fluid Mech. (1975), *vol.* 68, *part* 2, *pp.* 209–234
Printed in Great Britain

On the sedimentation of a sphere in a centrifuge

By ISOM H. HERRON,† STEPHEN H. DAVIS AND FRANCIS P. BRETHERTON‡

Department of Mechanics and Materials Science, The Johns Hopkins University, Baltimore, Maryland 21218

(Received 20 February 1974 and in revised form 22 July 1974)

The flow field about a small, slowly sedimenting particle in a centrifuge is examined using matched asymptotic expansions. The near field is dominated by Stokes flow while in the far field a non-axisymmetric cubical conical structure (a viscously modified Taylor column) is found. This far field induces a Coriolis modification in the near field leading to Coriolis corrections to the Stokes drag law. The Coriolis modification of the predicted molecular weight (if the particle were a molecule) of a small particle is calculated. The analysis is applied to an unbounded fluid as well as to a fluid bounded between parallel plates oriented normal to the rotation vector. In the latter case the governing equations for the rotating fluid are posed as a self-adjoint system of partial differential equations and solved using (symmetric) Green's matrices.

1. Introduction

The centrifuge is one of the tools used to infer molecular weights. One method, called equilibrium sedimentation, involves the spinning of the solution at low rotation rates. The rate must be sufficiently low so that sedimentation is balanced by back diffusion. The second main approach, called velocity sedimentation, involves large rotation rates. The rationale for the method is discussed in various sources. Bowen (1970), for example, examined the forces on a single particle, assuming instantaneous equilibrium (also see Schachman 1959, chaps. 4 and 6). The force balance considered is purely radial: a Stokes drag balances the local gravity, the centrifugal force. It seems clear, however, that, in a rotating system, non-radial forces must exist. A particle more dense than its fluid surroundings should not sediment radially but should be deflected by the presence of the Coriolis acceleration. Hence, even if a local equilibrium is assumed (this seems satisfactory if the drift speed is small enough), both radial and azimuthal forces are present and hence at least two components in the force balance are necessary.

The present study seeks to modify the Stokes drag law for an isolated spherical particle in an incompressible Newtonian fluid to account for the presence of rotation. Sharp & Beard (1950) and Cheng & Schachman (1955) have used

† Present address: Department of Mathematics, Howard University, Washington, D.C. 20059.

‡ Present address: National Center for Atmospheric Research, Boulder, Colorado 80302.

centrifugation to test the validity of the Stokes drag law but did not use rotational corrections. Langford (1968) sought to isolate the neglected dynamical effects inherent in the creeping-flow approximation but his approach was not fully consistent. He used an empirical drag relation that accounted for inertial but not Coriolis effects. Berman (1966) predicted a Coriolis deflexion of a particle influenced by Stokes drag but his effects were $O(T)$ as $T \to 0$, where

$$T = 2\Omega a^2/\nu.$$

The Taylor number T contains the constant rotation rate Ω of the centrifuge, the radius a of the particle and the kinematic viscosity ν of the surrounding fluid. The results of the present theory give corrections $O(T^{\frac{1}{2}})$, which for small values of T dominate those of Berman. If $\mathbf{U}$ is the drift velocity of the particle in the frame of reference rotating with the centrifuge, the Stokes drag is a force $-D\mathbf{U}$, where $D = 6\pi a\rho\nu$. The sum of this drag and the net centrifugal force $\mathbf{F}$ must vanish, hence determining $\mathbf{U}$ in terms of $\mathbf{F}$. When Coriolis corrections are included, the drag force is no longer parallel to $\mathbf{U}$, but has a transverse component $\frac{3}{5}(\frac{1}{2}T)^{\frac{1}{2}}DU$ in the direction of $\mathbf{U}\times\mathbf{\Omega}$. Since the net centrifugal force, which drives the motion, is radially outwards from the rotation axis, the particle does not move precisely in this direction but is deflected through an angle

$$\theta = -\tfrac{3}{5}(\tfrac{1}{2}T)^{\frac{1}{2}} + O(T)$$

and follows a spiral that makes a constant angle with the local radius vector in the plane normal to $\mathbf{\Omega}$.

This theory is linear in the particle speed $\mathbf{U}$ and the inertial terms $D\mathbf{u}/Dt$ in the equations of motion of the fluid are everywhere completely neglected compared with the viscous and Coriolis terms. In addition the particle radius a is supposed small compared with the Ekman length $(\nu/2\Omega)^{\frac{1}{2}}$, so that the Coriolis force causes only a small modification of the Stokes flow near the sphere. At distances comparable with or greater than the Ekman length, on the other hand, the flow regime is completely altered and matched asymptotic expansions in powers of $T^{\frac{1}{2}}$ are used to relate it to the Stokes flow. The governing equations near the sphere have a *formal* $O(T)$ Coriolis term. However, the presence of the sphere drastically alters the flow far from the sphere and this far-field correction induces an $O(T^{\frac{1}{2}})$ correction near the sphere (through a matching condition). This outer flow perceives the particle as being a point force at $O(T^{\frac{1}{2}})$, so that the particle shape and orientation do not affect this Coriolis correction. In the non-spherical case, the drift velocity $\mathbf{U}$ may not be parallel to the centrifugal force $\mathbf{F}$ on the particle even if the Coriolis forces are neglected, so that the Stokes drag coefficient D may have to be replaced by a drag tensor whose elements depend on the particle orientation. A complete analysis is complicated but, as argued above, it is easily seen that the *additional* effect of the Coriolis forces to $O(T^{\frac{1}{2}})$ is still to cause a lateral drift $(F/10\pi\rho\nu)(\Omega/\nu)^{\frac{1}{2}}\mathbf{t}$, where $\mathbf{t}$ is a unit vector in the direction of $\mathbf{F}\times\mathbf{\Omega}$, regardless of size or shape.

The Coriolis modified drag law is used to find the modification of the predicted molecular weight of *isolated* particles in a centrifuge compared with that of the

Stokes drag law. It is found that within the limits of the theory it is possible for the modification to lead to 10 % changes.

Fujita (1962, p. 8) mentions the Coriolis effect but dismisses it on the basis of the work of Hooyman *et al.* (1953), who consider a distribution of particles in a thermodynamic analysis. They find that there is no explicit appearance of the Coriolis effect in the entropy production and ignore the implicit effect. In fluid mechanical terms this means that in the kinetic energy equation of the fluid formed by dotting the Navier–Stokes equations with the velocity vector, the Coriolis force does not appear explicitly since it is a conservative field. However, the reduced pressure is affected, so that the force exerted by the fluid on the particle is modified. This is precisely the effect being calculated in the present work.

The drastically altered far field exhibits a non-axisymmetric cubical conical structure above and below the particle which is a viscosity-modified version of a Taylor column. The structure is given in detail.

This analysis has some similarities to that of Childress (1964), except that the particle motion is perpendicular rather than parallel to the rotation axis. These approximations are self-consistent within the Reynolds radius ν/U if, besides the Taylor number, the Rossby number $U(\nu\Omega)^{-\frac{1}{2}}$ is also much less than unity. In practice, both these conditions are usually satisfied if the particle is small enough.

If the fluid is bounded by a pair of rigid parallel walls normal to $\mathbf{\Omega}$ with the particle in the midplane, the deflexion angle θ is diminished to an extent which varies monotonically with the wall separation (at least correct to $O(T^{\frac{1}{2}})$). Besides the wall effect, this part of the analysis shows that the set of equations for the vertical velocity and vertical vorticity for linearized rotating flows can be posed as a self-adjoint system which can be solved using (symmetric) Green's matrices. This approach holds for quite general rotating fluid systems and so should be useful in other contexts.

2. The slow motion of a sphere in an unbounded rotating fluid

2.1. *The governing system*

Let us consider an infinite homogeneous body of fluid of kinematic viscosity ν and density ρ which rotates at constant angular velocity $\mathbf{\Omega}$. A sphere of radius a is translating unidirectionally orthogonal to $\mathbf{\Omega}$ with speed U.

To describe the motion a Cartesian co-ordinate system (x^*, y^*, z^*) is erected whose orthonormal unit vectors ($\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$) are as follows. The system rotates with angular speed $\Omega = |\mathbf{\Omega}|$ about the z^* axis $\mathbf{h}_3$, so that $\mathbf{\Omega} = \Omega\mathbf{h}_3$, and translates with the speed U. The direction of this unidirectional motion is defined to be in the negative $\mathbf{h}_1$ direction; hence, with respect to the rotating translating co-ordinate system fixed to the sphere, the fluid far from the sphere appears to be translating in the positive $\mathbf{h}_1$ direction.

The phenomena to be described are governed by the steady Navier–Stokes equations linearized in this co-ordinate system:

$$0 = -\nabla^* P^* - \Omega\mathbf{h}_3 \times \mathbf{v}^* + \nu\nabla^{*2}\mathbf{v}^*, \tag{2.1}$$

where the reduced pressure P^* satisfies

$$P^* = p^* - \tfrac{1}{2}\rho\Omega^2(x^{*2}+y^{*2})$$

and p^* is the pressure. The continuity equation for an incompressible fluid has the form

$$\nabla^* . \mathbf{v}^* = 0. \tag{2.2}$$

The appropriate boundary condition on the sphere is that

$$\mathbf{v}^* = 0 \quad \text{on} \quad r^* = a. \tag{2.3a}$$

Far from the sphere

$$\mathbf{v}^* \to U\mathbf{h}_1 \quad \text{as} \quad r^* \to \infty. \tag{2.3b}$$

The neglect of the time derivatives and the loss of advective terms demand some explanation. In steady-state Stokes–Oseen theory the Reynolds radius $r_R = \nu/U$ is the distance from the particle (in an order-of-magnitude sense) at which the inertial terms $\mathbf{v}^*.\nabla^*\mathbf{v}^*$ become comparable with the viscous terms. If the Reynolds number is small, $r_R \gg a$, there is a region around the particle where the viscous terms dominate. In a rotating fluid the Ekman radius $r_E = (\nu/2\Omega)^{\frac{1}{2}}$ is the distance at which the Coriolis forces first become comparable with the viscous ones. If $r_R \gg r_E \gg a$, the Stokes flow around the sphere is modified by the Coriolis forces, rather than the inertial ones. The latter become important only at very large distances, and the dominant corrections to the Stokes drag law are provided by the Coriolis forces, not the inertial ones (see § 4).

A final assessment of the importance of time derivatives $\partial\mathbf{v}^*/\partial\mathbf{t}^*$ (in a frame of reference moving with the particle) can only be made after solution of the equation balancing the drag forces and the net centrifugal force $\mathbf{F}$, and computation of the particle path. As the particle moves radially, $\mathbf{F}$ will change with time, and so will $\mathbf{U}$. However, the time scale for this is R/U, where R is the distance from the axis of rotation. If $U/R\Omega \ll 1$, then $|\partial\mathbf{v}^*/\partial t^*| \ll |2\mathbf{\Omega}\times\mathbf{v}|$ everywhere and neglect in (2.1) would seem justified.

The following parameter values are taken from an experiment by Sharp & Beard (1950) to determine the validity of the Stokes law. The acceleration used in the centrifuge was $\Omega^2\bar{r} = 2890g \approx 2{\cdot}83\times10^6\,\text{cm/s}^2$, $\bar{r} = 6{\cdot}5\,\text{cm}$ was the mean radius of rotation of the rotor and $\Omega = 6{\cdot}6\times10^2\,\text{rad/s}$ was the actual rotation rate. The density difference was $\bar{\rho}-\rho = 0{\cdot}02\,\text{gm/cm}^3$, where $\bar{\rho}$ is the density of the polystyrene latex (PSL) particles used and ρ is the density of the fluid. The particle size was $a \approx 1300\,\text{Å} = 1{\cdot}3\times10^{-5}\,\text{cm}$, and was to be determined accurately by the experiment. This particle size is only eight times those of some plant viruses. The force 'balance' between centrifugal and drag forces (Bowen 1970) can be used to approximate a typical drift speed U:

$$U = \frac{\Omega^2\bar{r}(\bar{\rho}-\rho)\times\frac{4}{3}\pi a^2}{6\pi\rho\nu a} = \frac{2}{9}\frac{\Omega^2\bar{r}(\bar{\rho}-\rho)a^2}{\rho\nu} = 2{\cdot}1\times10^{-4}\,\text{cm/s}. \tag{2.4}$$

Then, the Reynolds radius $r_R = 47$ cm and the Ekman radius $r_E = 2{\cdot}7\times10^{-3}\,\text{cm}$. The Taylor number $T = 2\Omega a^2/\nu = (a/r_E)^2 = 2{\cdot}3\times10^{-5}$ and for this experiment even $T^{\frac{1}{2}}$, the appropriate small parameter, was still less than 0·005. Coriolis corrections in this case were then probably of slight practical importance.

On the other hand, for some industrial applications the particle sizes of interest may be as large as 10^{-3} cm. For the same values of the other parameters, the drift speed U is now closer to 1 cm/s and hence $r_R = 10^{-2}$. For such larger particles the hierarchy of length scales is still retained. Then, $T^{\frac{1}{2}} \approx 0{\cdot}36$ and the deflexion angle θ is substantial. For still larger particles an expansion in powers of $T^{\frac{1}{2}}$ is of dubious validity. Childress (1964) defined a parameter

$$\alpha = T\,Re^{-2} = (2\Omega r_E/U)^2,$$

where $Re = Ua/\nu$ is the particle Reynolds number. In terms of these parameters, the regime of interest is $\alpha \to \infty$ and, in the case of particles of radius 10^{-3} cm, the worst case for the present theory leads to a value $\alpha = 8{\cdot}3$, which according to his equation (20a) gives an error in the order-one drag correction of 7 parts in 332. Thus it is felt that, for motion in a centrifuge, advection need not be considered in a first approximation.

It is now necessary to identify non-dimensional scales for the problem. As in the case of the slow motion of a sphere in a non-rotating fluid, two possible non-dimensional scales for asymptotic expansions can be identified. They will be called *inner* and *outer* scales, and the next two subsections will be devoted to defining them. The only portion of the inertial force to be retained in the analysis will be the Coriolis force contribution.

2.2. *The inner equations*

In a neighbourhood of the sphere, the pertinent length scale must be based on the size of the sphere. Non-dimensionalize by writing

$$\mathbf{v}^* = U\mathbf{v}, \quad \mathbf{r}^* = a\mathbf{r}, \quad P^* = \rho\nu UP/a. \qquad (2.5\,a,b,c)$$

The inner equations become

$$0 = -\nabla P - T\mathbf{h}_3 \times \mathbf{v} + \nabla^2\mathbf{v}, \qquad (2.6\,a)$$

$$0 = \nabla . \mathbf{v}, \qquad (2.6\,b)$$

where

$$T = 2\Omega a^2/\nu. \qquad (2.6\,c)$$

The scaled boundary conditions are

$$\mathbf{v} = 0 \quad \text{on} \quad r = 1 \qquad (2.6\,d)$$

and

$$P \to 0, \quad \mathbf{v} \to \mathbf{h}_1 \quad \text{as} \quad r \to \infty. \qquad (2.6\,e)$$

2.3. *The outer equations*

The study of creeping flows in non-rotating systems (Kaplun & Lagerstrom 1957) has shown that, if the radius a is used as the length scale, non-uniformities in the approximate solution occur in the far field. A different far-field length scale must be found.

A formal search for outer variables is begun as follows. In the outer region, the length scale is the Ekman radius. Write

$$\mathbf{r}' = T^{\frac{1}{2}}\mathbf{r}, \quad \mathbf{v}' = \mathbf{v}, \quad P' = T^{-\frac{1}{2}}P. \qquad (2.7\,a,b,c)$$

The outer equations then have the form

$$0 = -\nabla' P' - \mathbf{h}_3 \times \mathbf{v}' + \nabla'^2 \mathbf{v}', \qquad (2.8\,a)$$

$$0 = \nabla' . \mathbf{v}'. \qquad (2.8\,b)$$

The appropriate boundary conditions must express the fact that the free stream is approached far from the body: i.e.

$$\mathbf{v}' \to \mathbf{h}_1 \quad \text{as} \quad r' \to \infty. \qquad (2.8\,c)$$

The outer regime is characterized by a formal retention of the full linearized momentum balance: pressure-gradient, Coriolis and viscous terms. As $T \to 0$, $\mathbf{r}' \to 0$, so that in the outer region the sphere is seen as a single point $\mathbf{r}' = 0$. This point affects the flow field by exerting point forces (and higher-order contributions) on the fluid. A formal derivation using a multipole expansion is given in appendix A. These point singularities appear on the left-hand side of (2.8a). The leading term is $O(T^{\frac{1}{2}})$ in outer variables and is a point force called a Stokeslet. The left-hand side of (2.8a) corresponding to the Stokeslet is $6\pi\delta(\mathbf{r}')$ at $O(T^{\frac{1}{2}})$ as given in (A 8). The solutions to the $O(1)$ and $O(T^{\frac{1}{2}})$ problems will be obtained and shown to satisfy the matching conditions of Kaplun & Lagerstrom (1957).

2.4. *The inner expansion at* $O(1)$

In spite of the fact that the Taylor number appears in the differential equation to the first power, the matching conditions suggest an $O(T^{\frac{1}{2}})$ correction to the $O(1)$ terms. Hence, the formal inner expansion is obtained by substituting

$$\mathbf{v} = \mathbf{v}^{(0)} + T^{\frac{1}{2}}\mathbf{v}^{(1)} + \ldots, \qquad (2.9\,a)$$

$$P = P^{(0)} + T^{\frac{1}{2}}P^{(1)} + \ldots \qquad (2.9\,b)$$

into system (2.6). The $O(1)$ terms are as follows:

$$0 = -\nabla P^{(0)} + \nabla^2 \mathbf{v}^{(0)}, \quad 0 = \nabla . \mathbf{v}^{(0)}. \qquad (2.10\,a,b)$$

One solution of (2.10) is the Stokeslet. The Stokeslet exerts a drag force of magnitude 6π in the x' direction. The *unit* Stokeslet is defined by

$$\mathbf{S} = S_u \mathbf{h}_1 + S_v \mathbf{h}_2 + S_w \mathbf{h}_3, \qquad (2.11\,a)$$

where

$$S_u = \frac{1}{8\pi}\left[x\frac{\partial}{\partial x}\left(\frac{1}{r}\right) - \frac{1}{r}\right], \qquad (2.11\,b)$$

$$S_v = \frac{1}{8\pi}\left[x\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\right], \qquad (2.11\,c)$$

$$S_w = \frac{1}{8\pi}\left[x\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right], \qquad (2.11\,d)$$

and

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}}.$$

The corresponding vertical vorticity component of the Stokeslet is

$$S_\zeta = \frac{1}{4\pi}\left[\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\right]. \tag{2.11 e}$$

The complete solution of (2.10) satisfying boundary conditions (2.6 d, e) is given by

$$\mathbf{v} = \mathbf{h}_1 + 6\pi\mathbf{S} - \tfrac{1}{4}\nabla\left(\frac{\mathbf{h}_1 \cdot \mathbf{r}}{r^3}\right). \tag{2.11 f}$$

The last term represents a dipole which decays like r^{-3} as $r \to \infty$ whereas $|\mathbf{S}| \sim r^{-1}$. Hence, the most important terms for matching to the outer solution are $\mathbf{h}_1$ and $6\pi\mathbf{S}$. The strength of the Stokeslet is determined *solely* by the force $\mathbf{F}$ that the particle exerts on the fluid and is *independent* of the particle *shape* or *orientation* even if it is tumbling. This well-known result follows by recalling that in Stokes flow the viscous stresses can redistribute external forces but cannot balance them. Thus the stress integrated over the surface of any sphere enclosing the origin must equal $-\mathbf{F}$ independent of its radius. At large distances, only the Stokeslet is important since its stresses are $O(r^{-2})$.

2.5. *The outer expansion at* $O(1)$ *and* $O(T^{\frac{1}{2}})$

Let us look for solutions of the form

$$v_1'(\mathbf{r}') = v_1'^{(0)}(\mathbf{r}') + T^{\frac{1}{2}} v_1'^{(1)}(\mathbf{r}') + \ldots, \tag{2.12 a}$$

$$P'(\mathbf{r}') = P'^{(0)}(\mathbf{r}') + T^{\frac{1}{2}} P'^{(1)}(\mathbf{r}') + \ldots. \tag{2.12 b}$$

The expansions (2.12) are substituted into the form of (2.10) for forcing by an $O(T^{\frac{1}{2}})$ Stokeslet, i.e. the left-hand side of (2.10 a) is given by $T^{\frac{1}{2}} 6\pi\delta_{i1}\,\delta(\mathbf{r}')$. At $O(1)$ there is a geostrophic balance:

$$v_1'^{(0)} = \delta_{i1}, \quad P'^{(0)} = -y'. \tag{2.13 a, b}$$

At $O(T^{\frac{1}{2}})$ the equations are

$$\nabla'^2 v_i'^{(1)} - \epsilon_{i3k} v_k'^{(1)} - \partial P'^{(1)}/\partial x_i' = 6\pi\,\delta_{i1}\,\delta'(\mathbf{r}') \tag{2.14 a}$$

and

$$\partial v_i'^{(1)}/\partial x_i' = 0. \tag{2.14 b}$$

The appropriate boundary equations are

$$v_i'^{(1)} \to 0 \quad \text{as} \quad r' \to \infty. \tag{2.14 c}$$

The superscripts will henceforth be deleted.

Let us denote the Fourier transform of $\mathbf{v}'(\mathbf{r}')$ by $\hat{\mathbf{v}}'(\mathbf{k})$, so that

$$\mathbf{v}'(\mathbf{r}') = \frac{1}{8\pi^3}\iiint_{-\infty}^{\infty} \hat{\mathbf{v}}'(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}'}\, d\mathbf{k}, \tag{2.15}$$

where $\mathbf{k} = (k, l, m)$ is the wavenumber vector. If equations of the form (2.15) are substituted into (2.14 a, b), it follows that

$$\hat{u}' = -6\pi\,(m^2 + l^2)\,|\mathbf{k}|^2/(|\mathbf{k}|^6 + m^2), \tag{2.16 a}$$

$$\hat{v}' = 6\pi(kl\,|\mathbf{k}|^2 + m^2)/(|\mathbf{k}|^6 + m^2), \tag{2.16 b}$$

$$\hat{w}' = 6\pi(km\,|\mathbf{k}|^2 - ml)/(|\mathbf{k}|^6 + m^2). \tag{2.16 c}$$

The transform of the z component of vorticity is given by

$$\hat{\zeta}' = 6\pi(il\,|\mathbf{k}|^4 + im^2k)/(|\mathbf{k}|^6 + m^2). \tag{2.16d}$$

Although integrals like (2.15) are not absolutely convergent near the origin these expressions may be interpreted as generalized functions (Gel'fand & Shilov 1964). Consider the inverse transform of, say, the y component of velocity:

$$v'(\mathbf{r}') = \frac{3}{4\pi^2}\int \frac{kl\,|\mathbf{k}|^2 + m^2}{|\mathbf{k}|^6 + m^2}\, e^{i\mathbf{k}\cdot\mathbf{r}'}\, d\mathbf{k}. \tag{2.17}$$

(Henceforth, a single integral sign without limits denotes integration over all space.) This integral contains all the information about the y component of the outer flow field to $O(T^{\frac{1}{2}})$ but is not convergent for $\mathbf{r}' = 0$. Since the Stokeslet serves as a force singularity at $\mathbf{r}' = 0$, if viewed from the outer region, it seems reasonable to attempt to evaluate the integral in (2.17) minus the lateral Stokeslet velocity. The remaining difference *is* convergent at $\mathbf{r}' = 0$. The result is

$$v' - 6\pi S'_v\,|_{\mathbf{r}'=0} = \frac{3}{4\pi^2}\int \frac{m^2|\mathbf{k}|^4 - klm^2}{|\mathbf{k}|^4(|\mathbf{k}|^6 + m^2)}\, d\mathbf{k},$$

where

$$6\pi S'_v = \tfrac{3}{4}v'\frac{\partial}{\partial y'}\left(\frac{1}{r'}\right) = \frac{3}{4\pi^2}\int \frac{kl}{|\mathbf{k}|^4}\, e^{i\mathbf{k}\cdot\mathbf{r}'}\, d\mathbf{k}.$$

The integral can be evaluated using spherical co-ordinates. The result is

$$v' - 6\pi S'_v\,|_{\mathbf{r}'=0} = 3/5\sqrt{2}. \tag{2.18a}$$

The components u', w' and ζ' follow in a similar manner:

$$u' - 6\pi S'_u\,|_{\mathbf{r}'=0} = \frac{3}{4\pi^2}\int \frac{m^2(m^2 + l^2)}{|\mathbf{k}|^4(|\mathbf{k}|^6 + m^2)}\, d\mathbf{k} = \frac{5}{7\sqrt{2}}, \tag{2.18b}$$

$$w' - 6\pi S'_w\,|_{\mathbf{r}'=0} = -\frac{3}{4\pi^2}\int \frac{ml\,|\mathbf{k}|^4 + km^3}{|\mathbf{k}|^4(|\mathbf{k}|^6 + m^2)}\, d\mathbf{k} = 0, \tag{2.18c}$$

$$\zeta' - 6\pi S'_\zeta\,|_{\mathbf{r}'=0} = \frac{3i}{4\pi^2}\int \frac{m^2k\,|\mathbf{k}|^2 - lm^2}{|\mathbf{k}|^2(|\mathbf{k}|^6 + m^2)}\, d\mathbf{k} = 0. \tag{2.18d}$$

Further terms in the near-field outer expansion about $\mathbf{r}' = 0$ may be extracted and are as follows:

$$u'(\mathbf{r}') \sim 1 + T^{\frac{1}{2}}\left[\frac{3x'}{4}\frac{\partial}{\partial x'}\left(\frac{1}{r'}\right) - \frac{3}{4r'} + \frac{5}{7\sqrt{2}} + \frac{2}{195\sqrt{2}}\left(-\tfrac{7}{2}x'^2 - \tfrac{11}{2}y'^2 - \tfrac{55}{4}z'^2\right)\right] + O(T), \tag{2.19a}$$

$$v'(\mathbf{r}') \sim T^{\frac{1}{2}}\left[\frac{3x'}{4}\frac{\partial}{\partial y'}\left(\frac{1}{r'}\right) + \frac{3}{5\sqrt{2}} - \frac{3}{16}\left(r' + \frac{z'^2}{r'}\right) + \frac{4x'y'}{195\sqrt{2}} + O(r'^3)\right] + O(T), \tag{2.19b}$$

$$w'(\mathbf{r}') \sim T^{\frac{1}{2}}\left[\frac{3x'}{4}\frac{\partial}{\partial z'}\left(\frac{1}{r'}\right) + \frac{3}{16}\left(\frac{y'z'}{r'}\right) + \frac{2x'z'}{39\sqrt{2}} + O(r'^3)\right] + O(T), \tag{2.19c}$$

$$\zeta'(\mathbf{r}') \sim T^{\frac{1}{2}}\left[\frac{3}{2}\frac{\partial}{\partial y'}\left(\frac{1}{r'}\right) - \frac{3x'}{16}\left(\frac{x'^2 + y'^2}{r'^2}\right) + \frac{2y'}{15\sqrt{2}} + O(r'^2)\right] + O(T). \tag{2.19d}$$

The expansions (2.19) represent the near-field expansions as $\mathbf{r}' \to 0$. It will be shown in the next section that these components satisfy the equations of motion and that they match the asymptotic expansions in inner variables for flow near the sphere.

2.6. *Matching and drag modification*

Comparison of (2.19) and (2.11*f*), bearing in mind that $\mathbf{r}' = T^{\frac{1}{2}}\mathbf{r}$, shows that these solutions correspond in the region $r' \to 0$, $r \to \infty$, in so far as the first two terms in each are alternative representations of the same function. The uniform stream $\mathbf{h}_1$ appears in both. The Stokeslet field in the inner region decays like r^{-1} as $r \to \infty$ and so appears as a $T^{\frac{1}{2}}r'^{-1}$ singularity at the origin of the outer expansion. The dipole field $\nabla(\mathbf{h}_1 . \mathbf{r}/r^3)$, on the other hand, is comparable in magnitude with the Stokeslet field near the sphere where $r \sim 1$, but would correspond to a term $O(T^{\frac{3}{2}}r'^{-3})$ in (2.11*f*). This is negligible to $O(T^{\frac{1}{2}})$. In a similar way the terms $O(T^{\frac{1}{2}}r'^2)$ in (2.19) would be $O(T^{\frac{3}{2}})$ in the inner region and are absent from (2.11*f*). However, the uniform stream $T^{\frac{1}{2}}(5/7\surd 2,\ 3/5\surd 2,\ 0)$ in the neighbourhood of the origin in the outer expansion induces an $O(T^{\frac{1}{2}})$ correction to the inner Stokes flow. This correction is easily estimated, for it is associated only with a change in the boundary conditions as $r \to \infty$, and that change corresponds simply to altering the direction and magnitude of the incident uniform stream. The associated drag on the sphere is then

$$\mathbf{D} = 6\pi\{(1,0,0) + T^{\frac{1}{2}}(5/7\surd 2,\ 3/5\surd 2,\ 0) + O(T)\}. \tag{2.20}$$

This result is the central one of this paper and it shows clearly why the dominant correction due to the rotation is $O(T^{\frac{1}{2}})$, arising predominantly from effects far from the sphere itself at distances comparable with the Ekman radius. This is in contrast to that effect of rotation originating near the sphere which is $O(T)$.

2.7. *Far-field structure*

At distances large compared with the Ekman radius, the qualitative behaviour of the velocity field is governed in a gross sense by a geostrophic balance between Coriolis and pressure forces. Hence, there is a tendency for the length scale in the z direction to be much longer than that in the x and y directions, and the x, y velocity divergence is small. These features (characteristic of what is generally known as a Taylor column) are *local* consequences of the basic rotation. For the *global* structure, viscous forces must be significant (since the inertial terms have been entirely neglected). The outcome is a 'cone' that is most simply expressed in terms of the variables z', $\eta = (x'^2+y'^2)^{\frac{1}{2}}/|z'|^{\frac{1}{3}}$ and $\alpha = \tan^{-1}(x'/y')$. This cubical cone is not axisymmetric; the dominant terms at large $|z'|$ have w' proportional to $\cos\alpha$, u' proportional to $\cos 2\alpha$ and v' proportional to $\sin 2\alpha$. This asymmetry reflects that in the Stokeslet singularity which forces the whole motion. As $|z'| \to \infty$, the cone decays to zero; $w' \sim |z'|^{-1}$ for fixed η and α. The surfaces η = constant for the large values of r' being considered here are greatly elongated in the z' direction. Alternatively, if one looks in unscaled co-ordinates (x', y', z'), then the field appears *locally* cylindrical.

The structure can be obtained by examination of, say, the velocity component v'.

For convenience, let us write from (2.17)

$$v'_a(\mathbf{r}') = \frac{3}{4\pi^2}\int \frac{kl|\mathbf{k}|^2}{|\mathbf{k}|^6+m^2} e^{i\mathbf{k}.\mathbf{r}}\, d\mathbf{k} \tag{2.21 a}$$

and

$$v'_b(\mathbf{r}') = \frac{3}{4\pi^2}\int \frac{m^2}{|\mathbf{k}|^6+m^2} e^{i\mathbf{k}.\mathbf{r}}\, d\mathbf{k}, \tag{2.21 b}$$

when

$$v'(\mathbf{r}') = v'_a(\mathbf{r}') + v'_b(\mathbf{r}').$$

Introduce cylindrical co-ordinates

$$k = \lambda\cos\theta, \quad l = \lambda\sin\theta$$

and write

$$x'\cos\theta + y'\sin\theta = \sigma'\sin(\theta+\alpha).$$

Then from (2.21 a),

$$v'_a(\mathbf{r}') = \frac{3}{4\pi^2}\int_0^\infty d\lambda \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dm \frac{\lambda^3\sin\theta\cos\theta(\lambda^2+m^2)\exp[i\lambda\sigma'\sin(\theta+\alpha)]\, e^{im|z'|}}{(\lambda^2+m^2)^3+m^2}.$$

Poles of the integrand occur when

$$(\lambda^2+m^2)^3+m^2 = 0, \tag{2.22 a}$$

so that

$$v'_a(\mathbf{r}') = \frac{3}{4\pi^2}\int_0^\infty d\lambda \int_0^{2\pi} d\theta \exp[i\lambda\sigma'\sin(\theta+\alpha)] \times \left[2\pi i \sum_n \frac{\frac{1}{2}\lambda^3\, e^{im_n|z'|}\sin\theta\cos\theta\,(m_n^2+\lambda^2)}{m_n[3(\lambda^2+m_n^2)^2+1]}\right], \tag{2.22 b}$$

where the m_n are the roots of (2.22 a). When r' becomes large or when σ' and z' become large, the major contribution to the integrand in (2.22 b) occurs when $m_n(\lambda)|z'|$ (a complex quantity) has its minimum imaginary part, or perhaps where the denominator is small, or stationary, namely when

$$3(\lambda^2+m_n)^2+1 = 0.$$

The latter condition is incompatible with (2.22 a). Thus the minimum imaginary part occurs where λ is small. Then,

$$m_1(\lambda) \sim i\lambda^3 + O(\lambda^7),$$

$$m_2(\lambda) \sim (\tfrac{1}{2})^{\frac{1}{2}}(1+i) + O(\lambda^2),$$

$$m_3(\lambda) \sim -(\tfrac{1}{2})^{\frac{1}{2}}(1-i) + O(\lambda^2)$$

implies that $\exp[im_n(\lambda)|z'|]$ becomes exponentially small as $|z'| \to \infty$. Thus

$$v'_a \sim -\frac{3\sin 2\alpha}{4|z'|}\int_0^\infty \lambda^2 e^{-\lambda^3|z'|} J_2(\lambda\sigma')\, d\lambda + \ldots$$

$$= -\frac{3\sin 2\alpha}{4|z'|}\int_0^\infty s^2 e^{-s^3} J_2(\eta s)\, ds + O(|z'|^{-\frac{7}{3}}).$$

A similar examination shows that v'_b, which is independent of α, decays as $|z'|^{-\frac{5}{3}}$ and is thus smaller for large $|z'|$. The dominant terms are then as follows:

$$u' \sim \frac{3}{4|z'|}\left[\cos 2\alpha \int_0^\infty s^2 e^{-s^3} J_2(\eta s)\, ds - \int_0^\infty s^2 e^{-s^3} J_0(\eta s)\, ds\right], \qquad (2.23\,a)$$

$$v' \sim -\frac{3\sin 2\alpha}{4|z'|}\int_0^\infty s^2 e^{-s^3} J_2(\eta s)\, ds, \qquad (2.23\,b)$$

$$w' \sim \frac{3\cos\alpha}{2z'}\int_0^\infty s^2 e^{-s^3} J_1(\eta s)\, ds, \qquad (2.23\,c)$$

$$\zeta' \sim -\frac{3\cos\alpha}{2|z'|^{\frac{4}{3}}}\int_0^\infty s^2 e^{-s^3} J_1(\eta s)\, ds. \qquad (2.23\,d)$$

An analogous structure in two dimensions has been studied by Bretherton (1967). He solved the initial-value problem governing a circular cylinder towed through a fluid. His results show that the factor $\exp(-\lambda^3|z'|)$ is due to internal waves of wavenumber λ and zero frequency that propagate along the rotation axis and decay under the effect of viscosity. It is this factor that determines the similarity parameter η.

3. The slow motion of a sphere in a rotating fluid bounded between parallel planes

3.1. *The equations of motion, scaling and boundary conditions*

The problem of §2 is modified by the insertion of a pair of parallel plates perpendicular to $\mathbf{\Omega}$ and separated by a distance $2L$. The plates rotate with angular velocity $\mathbf{\Omega}$. A sphere of radius a moves at right angles to $\mathbf{\Omega}$ along the midplane. The same co-ordinate system as before is used, i.e. a Cartesian system that rotates with $\mathbf{\Omega}$ and translates with the sphere. The appropriate boundary conditions on the plates are that

$$\mathbf{v}^* = U\mathbf{h}_1 \quad \text{at} \quad z^* = \pm L. \qquad (3.1)$$

There are now three scales: a, L and $(\nu/\Omega)^{\frac{1}{2}}$. As before, it is assumed that the Reynolds radius $r_R = \nu/U$ is much greater than any of these. Define three non-dimensional parameters:

$$T = \frac{2\Omega a^2}{\nu}, \quad T_L = \frac{2\Omega L^2}{\nu}, \quad \left(\frac{T}{T_L}\right)^{\frac{1}{2}} = \frac{a}{L}. \qquad (3.2)$$

The inner problem is the same as that in the unbounded flow:

$$0 = -\nabla P - T\mathbf{h}_3 \times \mathbf{v} + \nabla^2 \mathbf{v}, \qquad (3.3\,a)$$

$$0 = \nabla . \mathbf{v}. \qquad (3.3\,b)$$

The inner expansion is also preserved:

$$\mathbf{v} = \mathbf{v}^{(0)}(\mathbf{r}) + T^{\frac{1}{2}}\mathbf{v}^{(1)}(\mathbf{r}) + \ldots, \qquad (3.3\,c)$$

with $\mathbf{r} = a^{-1}\mathbf{r}^*$ and $\mathbf{v}^{(0)}(\mathbf{r}) = 6\pi\mathbf{S}(\mathbf{r})$, where $\mathbf{S}$ represents a Stokeslet defined on $1 \leqslant r < \infty$.

The outer length scale L in this bounded problem is provided by the plate spacing. Here

$$\mathbf{r}_0 = L^{-1}\mathbf{r}^* = aL^{-1}\mathbf{r} = (T/T_L)^{\frac{1}{2}}\mathbf{r} \tag{3.4}$$

on $-\infty < x_0, y_0 < \infty$ and $-1 \leqslant z_0 \leqslant 1$. It is assumed that $a/L \ll 1$, so that $(T/T_L)^{\frac{1}{2}} \ll 1$.

Intermediate scales are determined by the ordering of a, L and $(\nu/\Omega)^{\frac{1}{2}}$. There must exist scaled radii $\mathbf{r}'$ and $\mathbf{r}''$ given by $\mathbf{r}' = T^{\frac{1}{2}}\mathbf{r}$ and $\mathbf{r}'' = T_L^{-\frac{1}{2}}\mathbf{r}$, which are not as 'far out' as $\mathbf{r}_0$. (This $\mathbf{r}'$ is the outer radial co-ordinate of the unbounded problem.) Moreover, if $T_L \gg 1$, then z_0 might have a boundary-layer scale, say $z_B = T_L^{\frac{1}{2}}(1 \pm z_0)$. The problem then falls into the province of almost-rigid rotations considered by Stewartson (1957), Greenspan (1968, p. 100) and Moore & Saffman (1969). If T_L is not large, then the 'boundary layers' at the plates will overlap with the $\mathbf{r}'$ and $\mathbf{r}''$ regions. This case can only be treated by solving the differential system for arbitrary T_L. This will be done later.

3.2. *The interior fluid motion for T_L large*

The components w' and ζ' are the most convenient to examine. Their governing equations can be obtained directly from system (2.14) and are as follows:

$$\begin{pmatrix} \nabla'^4 & -\partial/\partial z' \\ \partial/\partial z' & \nabla'^2 \end{pmatrix} \begin{pmatrix} w' \\ \zeta' \end{pmatrix} = -6\pi \begin{pmatrix} \delta'(x')\,\delta(y')\,\delta'(z') \\ \delta(x')\,\delta'(y')\,\delta(z') \end{pmatrix}. \tag{3.5a}$$

The boundary conditions are

$$\begin{pmatrix} \partial/\partial z' & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w' \\ \zeta' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{as} \quad r' \to \infty. \tag{3.5b}$$

The similarity variable in the outer region of the unbounded problem is $\eta = |z'|^{-\frac{1}{3}}\sigma'$. This form survives in the bounded problem with $\sigma' = T_L^{\frac{1}{2}}\sigma_0$ for outer horizontal scale σ_0. The scaling is consistent with matching requirements for the *slowest* decaying parts at large distances of $w'(\mathbf{r}')$ and $\zeta'(\mathbf{r}')$ in (2.23*c*, *d*), which are given by

$$w'(x_0, y_0, z_0) \sim \left\{\tfrac{3}{2}\cos\alpha \int_0^\infty s^2 e^{-s^3} J_1(\eta s)\, ds\right\} T_L^{-\frac{1}{2}} |z_0|^{-1} \operatorname{sgn} z_0, \tag{3.6a}$$

and

$$\zeta'(x_0, y_0, z_0) \sim \left\{-\tfrac{3}{2}\cos\alpha \int_0^\infty s^2 e^{-s^3} J_1(\eta s)\, ds\right\} T_L^{-\frac{2}{3}} |z_0|^{-\frac{4}{3}}. \tag{3.6b}$$

Hence, matching is possible using these scales if

$$w_0 = T_L^{\frac{1}{2}} w', \quad \zeta_0 = T_L^{\frac{2}{3}} \zeta'. \tag{3.6c}$$

In terms of these, (3.5) can be Fourier transformed in x_0 and y_0 as in definitions (2.15) and the result to $O(T_L^{\frac{1}{2}})$ is as follows:

$$\begin{pmatrix} \lambda^4 & -D_0 \\ D_0 & -\lambda^2 \end{pmatrix} \begin{pmatrix} \hat{w}_0 \\ \hat{\zeta}_0 \end{pmatrix} = -6\pi i \begin{pmatrix} k\delta'(z_0) & T_L^{-\frac{1}{3}} \\ l\delta(z_0) & \end{pmatrix}, \tag{3.7a}$$

where

$$D_0 = d/dz_0, \quad \lambda = (k^2 + l^2)^{\frac{1}{2}}.$$

The appropriate boundary condition is the familiar Ekman-layer suction, given as follows:

$$\hat{w}_0(\pm 1) = \mp 2^{-\frac{1}{2}} T_L^{-\frac{1}{6}} \hat{\zeta}_0(\pm 1). \tag{3.7b}$$

The second-order system (3.7) is well posed. The solution is obtained by finding the Green's matrix $\mathbf{G}(z, \xi)$ satisfying columnwise the differential equation

$$\begin{pmatrix} \lambda^4 & -D_0 \\ D_0 & -\lambda^2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \delta(z-\xi) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.8}$$

and boundary conditions (3.7 b):

$$\begin{pmatrix} \hat{w}_0 \\ \hat{\zeta}_0 \end{pmatrix} = 6\pi \int_{-1}^{1} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} -ik\delta'(\xi)\, T_L^{-\frac{1}{3}} \\ -il\delta(\xi) \end{pmatrix}$$

$$= 6\pi \begin{pmatrix} ikT_L^{-\frac{1}{3}} g_{11,\xi}(z_0, 0) - ilg_{12}(z_0, 0) \\ ikT_L^{-\frac{1}{3}} g_{21,\xi}(z_0, 0) - ilg_{22}(z_0, 0) \end{pmatrix}. \tag{3.9}$$

From continuity and the definition of $\hat{\zeta}_0$, the other velocity components are given by

$$\hat{u}_0(z_0) = \frac{l\hat{\zeta}_0 + T_L^{-\frac{1}{3}} kD_0 \hat{w}_0}{-i\lambda^2} \sim \frac{6\pi l^2}{\lambda^2} g_{22}(z_0, 0) + O(T_L^{-\frac{2}{3}}), \tag{3.10a}$$

$$\hat{w}_0(z_0) = \frac{k\hat{\zeta}_0 - T_L^{-\frac{1}{3}} lD_0 \hat{w}_0}{i\lambda^2} \sim -6\pi \frac{kl}{\lambda^2} g_{12}(z_0, 0) + 6\pi T_L^{-\frac{1}{3}} g_{21,\xi}(z_0, 0) + O(T_L^{-\frac{2}{3}}). \tag{3.10b}$$

The relevant terms from the Green's matrix are

$$g_{22}(z_0, 0) = -\lambda(\cosh \lambda^3 + 2^{-\frac{1}{2}} T_L^{-\frac{1}{6}} \lambda \sinh \lambda^3) [\cosh(\lambda^3(1-|z_0|)) + 2^{-\frac{1}{2}} T_L^{-\frac{1}{6}} \lambda \sinh(\lambda^3(1-|z_0|))]/[\sinh 2\lambda^3 + 2^{\frac{1}{2}} \lambda T_L^{-\frac{1}{6}} \cosh 2\lambda^3], \tag{3.11a}$$

$$g_{21,\xi}(z_0, 0) = \lambda^2 g_{22}(z_0, 0), \tag{3.11b}$$

$$g_{11,\xi}(z_0, 0) = \operatorname{sgn}(z_0)\, \lambda^2 (\cosh \lambda^3 + 2^{-\frac{1}{2}} T_L^{-\frac{1}{6}} \lambda \sinh \lambda^3) [\sinh(\lambda^3(1-|z_0|)) + \frac{2^{-\frac{1}{2}} T_L^{-\frac{1}{6}} \lambda \cosh(\lambda^3(1+|z_0|))]}{[\sinh 2\lambda^3 + 2^{\frac{1}{2}} \lambda T_L^{-\frac{1}{6}} \cosh 2\lambda^3]}, \tag{3.11c}$$

$$g_{12,\xi}(z_0, 0) = \lambda^{-2} g_{11,\xi}(z_0, 0). \tag{3.11d}$$

The contribution of the wall effect to the drag on the sphere and the velocity components at the edges of the Ekman layers can now be computed.

For the unbounded flow the far field is given by (2.23). Then the wall effect is found by writing u' and v' in $\mathbf{r}_0$ variables and *subtracting* them from the corresponding components of the outer flow.

The wall effect D'_x on the x component of the drag as $T_L \to \infty$ is given by

$$\lim_{\mathbf{r}_0 \to 0} [T_L^{-\frac{1}{3}} u_0(\mathbf{r}_0) - u'(\mathbf{r}_0)]:$$

$$D'_x \sim -\tfrac{3}{4} T_L^{-\frac{1}{3}} \int_0^\infty \frac{\lambda^2[e^{-2\lambda^3}(1 - 2^{\frac{1}{2}} \lambda T_L^{-\frac{1}{6}}) + 1]}{\sinh 2\lambda^3 + 2^{\frac{1}{2}} \lambda T_L^{-\frac{1}{6}} \cosh 2\lambda^3} d\lambda + O(T_L^{-\frac{7}{6}}). \tag{3.12a}$$

Similarly the wall effect D'_y on the y component is given by

$$\lim_{\mathbf{r}_0\to 0}\,[T_L^{-\frac{1}{2}}v_0(\mathbf{r}_0)-v'(\mathbf{r}_0)]:$$

$$D'_y \sim -3T_L^{-\frac{5}{6}}\int_0^\infty \frac{\lambda^4[e^{-2\lambda^3}(1-2^{\frac{1}{2}}T_L^{-\frac{1}{6}}\lambda)+1]\,dk}{\sinh 2\lambda^3+2^{\frac{1}{2}}\lambda T_L^{-\frac{1}{6}}\cosh 2\lambda^3}+O(T_L^{-\frac{3}{2}}). \qquad (3.12\,b)$$

At the edge of the boundary layers, $|z_0|\to 1$, the components are as follows:

$$u \simeq \tfrac{3}{2}T_L^{-\frac{1}{2}}\int_0^\infty \frac{\lambda^2(\cosh\lambda^3+2^{-\frac{1}{2}}T_L^{-\frac{1}{6}}\lambda\sinh\lambda^3)}{\sinh 2\lambda^3+2^{\frac{1}{2}}\lambda T_L^{-\frac{1}{6}}\cosh 2\lambda^3}[J_0(\lambda\sigma_0)-\cos 2\alpha J_2(\lambda\sigma_0)]\,d\lambda+O(T_L^{-\frac{5}{6}}), \qquad (3.13\,a)$$

$$v \simeq -\tfrac{3}{2}T_L^{-\frac{1}{2}}\int_0^\infty \frac{\lambda^2(\cosh\lambda^3+2^{-\frac{1}{2}}T_L^{-\frac{1}{6}}\lambda\sinh\lambda^3)}{\sinh 2\lambda^3+2^{\frac{1}{2}}\lambda T_L^{-\frac{1}{6}}\cosh 2\lambda^3}\sin 2\alpha J_2(\lambda\sigma_0)\,d\lambda+O(T_L^{-\frac{5}{6}}), \qquad (3.13\,b)$$

$$w \simeq \pm\tfrac{3}{2}T_L^{-\frac{2}{3}}\int_0^\infty \frac{\lambda^2(\cosh\lambda^3+2^{-\frac{1}{2}}T_L^{-\frac{1}{6}}\lambda\sinh\lambda^3)}{\sinh 2\lambda^3+2^{\frac{1}{2}}\lambda T_L^{-\frac{1}{6}}\cosh 2\lambda^3}\cos\alpha J_1(\lambda\sigma_0)\,d\lambda+O(T_L^{-1}). \qquad (3.13\,c)$$

The vertical component of velocity at the edge of the boundary layer is $O(T_L^{-\frac{1}{6}})$ smaller than the others. This is a consequence of the continuity equation and the condition that the boundary-layer vertical component vanish at the walls $|z_0| = 1$.

3.3. *The drag on the sphere for arbitrary T_L*

A typical component of the Stokeslet, say S_u, is

$$S_u = \frac{1}{8\pi}\left[x\frac{\partial}{\partial x}\left(\frac{1}{r}\right)-\frac{1}{r}\right]$$

based on the length scale a. In this inner domain, $S_u = O(1)$ and u has the form $u = 1+6\pi S_u+O(T^{\frac{1}{2}})$. S_u can be expressed in terms of outer variables using the length scale L:

$$S_{u_0} = \frac{1}{8\pi}\left[x_0\frac{\partial}{\partial x_0}\left(\frac{1}{r_0}\right)-\frac{1}{r_0}\right]\left(\frac{T}{T_L}\right)^{\frac{1}{2}}. \qquad (3.14)$$

Thus, to $O((T/T_L)^{\frac{1}{2}})$, the Stokeslet (3.14) does not vanish on $z_0 = \pm 1$ and so provides a boundary value $\mathbf{v}_0 = -6\pi\mathbf{S}$ on $z_0 = \pm 1$. This is consistent with the reflexion principle (Happel & Brenner 1965, p. 286).

The outer equations, characterized by the length scale L, can be written as follows:

$$\nabla_0^2 v_{0i}-T_L\epsilon_{i3k}v_{0k}-\partial P_0/\partial x_{0i} = 6\pi(T/T_L)^{\frac{1}{2}}\,\delta(\mathbf{r}_0)\,\delta_{i1}+O(T/T_L). \qquad (3.15)$$

In all of what follows, let us consider T_L as arbitrary but fixed. The expansion in this outer region has the form

$$v_{0i} \sim v_{0i}^{(0)}(\mathbf{r}_0)+T^{\frac{1}{2}}v_{0i}^{(1)}(\mathbf{r}_0)+\dots. \qquad (3.16)$$

If expansion (3.16) is inserted into (3.15), it follows that, at

$$v_{0i}^{(0)}(\mathbf{r}_0) = \delta_{i1}, \qquad (3.17)$$

and at $O(T^{\frac{1}{2}})$,

$$P_0^2 v_{0i}^{(1)}-T_L\epsilon_{i3k}v_{0k}^{(1)}-\partial P_0^{(1)}/\partial x_{0i} = 6\pi T_L^{-\frac{1}{2}}\,\delta(\mathbf{r}_0)\,\delta_{i1}, \qquad (3.18\,a)$$

with

$$\partial v_{0i}^{(1)}/\partial x_{0i} = 0. \qquad (3.18\,b)$$

Let us omit the subscripts and superscripts and understand that from now on only the $O(T_L^{\frac{1}{2}})$ outer system will be considered.

By Fourier transformation and elimination the system (3.18) can be written in terms of the transforms $\hat{w}$ and $\hat{\zeta}$ of the vertical velocity and vorticity as follows:

$$\mathbf{L}\mathbf{\Psi} = \mathbf{f},$$

where

$$\mathbf{L} = \begin{pmatrix} (d^2/dz^2-\lambda^2)^2 & -T_L\, d/dz \\ T_L\, d/dz & (d^2/dz^2-\lambda^2) \end{pmatrix} \tag{3.19a}$$

and

$$\mathbf{\Psi} = \begin{pmatrix} \hat{w} \\ \hat{\zeta} \end{pmatrix}, \quad \mathbf{f} = -i6\pi T_L^{-\frac{1}{2}} \begin{pmatrix} k\delta'(z) \\ l\delta(z) \end{pmatrix}.$$

The boundary conditions are

$$\mathbf{B}\mathbf{x}(\mathbf{\Psi}(\pm 1)) = 0, \tag{3.19b}$$

where

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{x}(\mathbf{\Psi}) = \begin{pmatrix} \hat{w} \\ \hat{w}' \\ \hat{w}'' \\ \hat{w}''' \\ \hat{\zeta} \\ \hat{\zeta}' \end{pmatrix}.$$

In appendix B, it is shown that system (3.19) is self-adjoint and hence has a symmetric Green's matrix $\mathscr{G}$ which satisfies

$$\mathbf{L}\mathscr{G} = \delta(z-\xi)\,\mathbf{I}$$

and has the form

$$\mathscr{G} = \begin{pmatrix} G_{11}(z,\xi;\lambda) & G_{12}(z,\xi;\lambda) \\ G_{21}(z,\xi;\lambda) & G_{22}(z,\xi;\lambda) \end{pmatrix}.$$

Each column of $\mathscr{G}$ treated as a vector satisfies the boundary conditions (3.19*b*). Such an approach has been used to compute wall effects in non-rotating flows by Cox & Brenner (1967).

Given $\mathscr{G}$, the solution of system (3.19*a*) can be written in the form (see appendix B)

$$\begin{aligned} \mathbf{\Psi}(z) &= \int_{-1}^{1} \mathscr{G}(z,\xi)\,\mathbf{f}(\xi)\,d\xi \\ &= 6\pi T_L^{-\frac{1}{2}} \int_{-1}^{1} \begin{pmatrix} G_{11}(z,\xi;\lambda) & G_{12}(z,\xi;\lambda) \\ G_{21}(z,\xi;\lambda) & G_{22}(z,\xi;\lambda) \end{pmatrix} \begin{pmatrix} -ik\delta'(\xi) \\ -il\delta(\xi) \end{pmatrix} d\xi \\ &= 6\pi T_L^{-\frac{1}{2}} \begin{pmatrix} ikG_{11,\xi}(z,0) - ilG_{12}(z,0) \\ ikG_{21,\xi}(z,0) - ilG_{22}(z,0) \end{pmatrix}. \end{aligned} \tag{3.20}$$

The lateral velocity v can be put in terms of $\hat{w}$ and $\hat{\zeta}$ and hence expressed in terms of the G_{mn} through (3.20) as follows:

$$v(x,y,z) = \frac{3T_L^{-\frac{1}{2}}}{2\pi} \iint_{-\infty}^{\infty} [k^2G_{21,\xi}(z,0;\lambda) + l^2G_{12,z}(z,0;\lambda) - klG_{11,\xi z}(z,0;\lambda) - klG_{22}(z,0;\lambda)]\, e^{i(kx+ly)} \frac{dk\,dl}{k^2+l^2}.$$

The symmetry of the Green's matrix (appendix B) gives

$$G_{mn}(z,\xi) = G_{nm}(\xi,z),$$

so that

$$\partial G_{12}(\xi,\xi)/dz = \partial G_{21}(\xi,\xi)/\partial \xi.$$

Hence, the lateral velocity at $z = 0$ can be written as follows:

$$v(x,y,0) = \frac{3T_L^{-\frac{1}{2}}}{2\pi}\iint_{-\infty}^{\infty} dk\,dl\left[G_{12,z}(0,0;\lambda) - \frac{kl}{k^2+l^2}G_{11,\xi z}(0,0;\lambda) - \frac{kl}{k^2+l^2}G_{22}(0,0;\lambda)\right]e^{i(kx+ly)}. \quad (3.21)$$

The integrand in (3.21) is divergent at $x = y = 0$ but can be evaluated as before by subtracting out the Stokeslet singularity. The value of $v - 6\pi S_v$ gives the correction to the lateral velocity due to both Coriolis and wall effects.

The appropriate Fourier-transformed Stokeslet components are given by

$$\hat{S}_w = \tfrac{1}{4}ikze^{-\lambda|z|}/\lambda, \quad \hat{S}_\zeta = \tfrac{1}{2}il\,e^{-\lambda|z|}/\lambda.$$

In terms of these, the value of $\hat{S}_v$ follows:

$$\hat{S}_v = \frac{kl}{k^2+l^2}\left[\frac{1}{2\lambda} - \frac{1}{4\lambda} + z\,\mathrm{sgn}\,z\right]e^{-\lambda|z|}. \quad (3.22)$$

This is to be evaluated at $z = 0$.

The departure of the lateral velocity from that due to the Stokeslet can be thought of as an expansion about $r = 0$:

$$v - 6\pi S_v|_{r=0} = \frac{3T_L^{-\frac{1}{2}}}{2\pi}\lim_{x,y\to 0}\iint_{-\infty}^{\infty} dk\,dl\frac{e^{i(kx+ly)}}{k^3+l^2} \times\left[G_{12,z}(0,0,\lambda) - \frac{il}{k^2+l^2}\left(G_{11,\xi z} - \frac{1}{4\lambda}\right) - \frac{kl}{k^2+l^2}\left(G_{22} + \frac{1}{2\lambda}\right)\right]. \quad (3.23)$$

Polar co-ordinates are introduced into (3.23) for both the physical and Fourier-transformed variables. The integral over angle can immediately be evaluated. The integrand at $\lambda = 0$ is finite.

The terms which are asymptotically $O(\lambda^{-1})$ as $\lambda \to \infty$ cancel and the integral (3.24) converges for all σ. Now let $\sigma \to 0$. The non-zero contribution to the integrand comes from the term $G_{12,z}(\lambda)$.

The perturbation of the x component of velocity at the origin may likewise be calculated and has the form

$$u - 6\pi S_u|_{r=0} = \frac{3T_L^{-\frac{1}{2}}}{2\pi}\iint_{-\infty}^{\infty}\left[\frac{-k^2}{k^2+l^2}\left(G_{11,\xi} - \frac{1}{4\lambda}\right) + \frac{l^2}{k^2+l^2}\left(G_{22}(\lambda) + \frac{1}{2\lambda}\right)\right]e^{i(kx+ly)}dk\,dl. \quad (3.24)$$

The wall effect on the drag on the sphere can be extracted by computing the drag in the limit of *zero* rotation. This case of a sphere in uniform motion between fixed parallel plates has been computed by Faxén (1922) using the method of reflexions. The drag is obtained here by examining a Stokeslet between two walls (cf. Blake 1971) through the use of (3.19) with $T_L = 0$.

	Rotational		Non-rotational	
$T_L^{\frac{1}{2}}$	D_y	D_x	L/a	D_x
∞	$3/5\sqrt{2}$ = 0·424	$5/7\sqrt{2}$ = 0·505	∞	0·000
10	0·417	0·470	10	0·100
9		0·464	9	0·112
8	0·416	0·459	8	0·126
7		0·454	7	0·143
6	0·412	0·444	6	0·167
5	0·403	0·430	5	0·201
4	0·378	0·414	4	0·251
3	0·321	0·422	3	0·335
2	0·228	0·531	2	0·502
1	0·116	1·008	1	1·004

TABLE 1. Wall corrections to the drag with and without rotation

The drag corrections for both $T_L \neq 0$ and $T_L = 0$ were evaluated numerically. The results in table 1 compare the $O(T_L^{\frac{1}{2}})$ drag modifications experienced by the sphere as a function of T_L. The limit $T_L \rightarrow \infty$ gives the unbounded case derived analytically in §2 while the non-rotating limit recovers the results of Faxén (1922). The lateral drag, purely a product of rotation, is a monotonic function of T_L. The translational drag has a minimum occurring near $T_L = 13$. From another point of view it may be seen from the last column that the 'non-rotating contribution' dominates this component until approximately $T_L = 13$.

4. Conclusions

In a centrifuge the effective buoyancy force (centrifugal force) on a heavy particle is directed radially outwards from the axis of rotation and has a magnitude that depends upon the particle position. The analysis considered the steady linearized equations for the flow of a homogeneous viscous liquid in a rotating translating reference frame centred on the particle. The neglect of the fluid and frame accelerations is consistent if the particle is small enough and the motions are slow (see §2). The first case considered is when the boundaries are of negligibly small importance.

Near the particle the Coriolis force is small compared with the viscous forces, and Stokes flow is the dominant first approximation (the inner solution). For a spherical particle the classical Stokes solution, the sum of a uniform stream U, a dipole and a Stokeslet, provides the leading term of an asymptotic expansion. The force balance on a spherical surface enclosing the particle requires that the strength of the Stokeslet equal $(\rho\nu a U)^{-1}$ times the force exerted by the particle on the fluid, which in turn is the net centrifugal force on the particle. The *formal* correction to the Stokes flow due to Coriolis terms is $O(T)$, where $T = 2\Omega a^2/\nu$. However, the analysis shows that the true correction is $O(T^{\frac{1}{2}})$, arising from a modification of the velocity U 'at infinity'. Viewed from a distance large compared with the particle radius a, the Stokes flow appears as a Stokeslet singularity in the radial direction. At distances comparable with the Ekman radius $(\nu/2\Omega)^{\frac{1}{2}}$

(the outer region), Coriolis terms are comparable with the viscous and pressure forces, and the flow field completely different from Stokes flow. The detailed solution is given in § 2.7. Near the origin (which embraces the whole inner region) this solution represents the uniform stream U, the Stokeslet singularity and an additional uniform stream $O(T^{\frac{1}{2}})$. The latter is a translation of a significant fraction of the fluid within the Ekman sphere (of radius $(\nu/2\Omega)^{\frac{1}{2}}$); it is partly in the direction of the free stream $\mathbf{U}$ and partly transverse to it, i.e., perpendicular to both $\mathbf{U}$ and $\mathbf{\Omega}$. In contrast to the Stokes region the net force associated with the viscous and pressure stresses across a surface enclosing the Ekman sphere approaches zero as the surface becomes large. The force applied to the fluid at the origin, the Stokeslet, is balanced within this volume by the Coriolis force associated with an integrated drift in the direction transverse to the Stokeslet. The magnitude and the sense of the additional uniform stream at the origin are consistent with the following argument.

The volume of the fluid participating in this drift is $O((\nu/2\Omega)^{\frac{3}{2}})$ and if an average drift velocity is denoted by $\bar{\mathbf{u}}_d = (\bar{u}_d, \bar{v}_d)$, the associated Coriolis force is

$$2\mathbf{\Omega} \times \bar{\mathbf{u}}_d \rho(\nu/2\Omega)^{\frac{3}{2}}.$$

This must equal the Stokeslet force $\mathbf{F}$. Hence

$$\bar{v}_d \sim \frac{F}{\rho\nu}\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}, \tag{4.1}$$

in the direction of $\mathbf{F} \times \mathbf{\Omega}$. For a sphere

$$\mathbf{F} = 6\pi\rho\nu a\mathbf{U},$$

so that the transverse fluid drift has magnitude

$$\bar{v}_d \sim T^{\frac{1}{2}}U. \tag{4.2}$$

This is the average value over a large volume. The full analysis shows that the additional transverse velocity v_d at the origin is

$$v_d = \frac{3}{5}\frac{F}{6\pi\rho\nu}\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}. \tag{4.3}$$

The longitudinal component of velocity (in the direction of the applied force $\mathbf{F}$) of a *Stokeslet* field is everywhere positive. However, (4.1) shows that this component of the integrated drift velocity over the Ekman region must be zero. Plausibly the additional effect of the Coriolis force above the Stokeslet field is in the opposite direction to the latter and has an average value $\bar{u}_d$ over the Ekman sphere such that the corresponding volume flux $\bar{u}_d(\nu/\Omega)^{\frac{3}{2}}$ is comparable to the Stokeslet flux $(F/\rho\nu)(\nu/\Omega)$ contained within the same region. Hence, the additional longitudinal velocity u_d at the origin, which should be comparable to $\bar{u}_d$, is

$$u_d \sim \frac{F}{\rho\nu}\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}, \tag{4.4}$$

which serves to augment **U**. The full analysis shows that

$$u_d = \frac{5}{7}\frac{F}{6\pi\rho\nu}\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}. \tag{4.5}$$

This additional velocity $\mathbf{u}_d$ at the origin appears from the inner region as a modification of the velocity 'at infinity', from **U** to $\mathbf{U}+\mathbf{u}_d$. The drag $\mathbf{F}_d$ on the sphere is thus corrected by a term $O(T^{\frac{1}{2}})$ and is no longer parallel to **U**. It is given by

$$\mathbf{F}_d = -6\pi\rho\nu a\{\mathbf{I} + (\tfrac{1}{2}T)^{\frac{1}{2}}\,\mathbf{\Delta}\}\,\mathbf{U} + O(T),$$

where

$$\mathbf{\Delta} = \begin{pmatrix} \frac{5}{7} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{5}{7} & 0 \\ 0 & 0 & \frac{4}{7} \end{pmatrix}. \tag{4.6}$$

The last column of $\mathbf{\Delta}$ is the result of Childress (1964), who considered a sphere moving along the rotation axis. The remaining results were obtained in §2. Note that, owing to the rotation, $\mathbf{\Delta}$ is asymmetric, so that a preferred direction of Coriolis deflexion emerges.

For a particle of arbitrary shape and orientation the Stokes drag coefficient might have to be replaced by a drag tensor, in which case (even in the absence of rotation) the motion is not radial. However, the *additional* motion associated with the Coriolis forces is still given by the formulae (4.3) and (4.5). This is easily seen because the Stokes flow around an arbitrary finite body may be represented at large distances as a superposition of a uniform stream, a Stokeslet and terms that decay more rapidly with distance. The strength of the Stokeslet depends only on the total force **F** exerted by the body on the fluid and is independent of its shape and orientation. For the outer solution obtained above only the Stokeslet enters, though, if carried to a higher approximation, the shape would be relevant (see appendix A).

The effect of rotation on a particle can be examined by writing down Newton's law for a mass point. It is consistent with the linearized analysis to assume small relative motions between the point and the fluid and slow changes in velocity. Hence, the force balance to $O(T^{\frac{1}{2}})$ takes the form

$$-\beta\{\mathbf{I} + (\tfrac{1}{2}T)^{\frac{1}{2}}\mathbf{\Delta}\}\begin{pmatrix} V_r \\ V_\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -r\Omega^2 - 2\rho_r V_\theta \Omega - r^{-1}\rho V_\theta^2 \\ 2\rho_r \Omega V_r + r^{-1}\rho_r V_r V_\theta \\ 0 \end{pmatrix}, \tag{4.7}$$

where the particle velocity is $(V_r, V_\theta, 0)$,

$$\beta = \frac{9\nu}{2a^2}\left(\frac{\rho}{\bar\rho - \rho}\right), \quad \rho_r = \frac{\rho}{\bar\rho - \rho},$$

and ρ and $\bar\rho$ are respectively the fluid and particle densities. If, in addition, $V_\theta \ll r\Omega$ and it is recognized that $\Omega/\beta = O(T)$, then the velocity components to $O(T^{\frac{1}{2}})$ can be written as follows:

$$V_r = (\Omega^2 r/\beta)\,(1 - \tfrac{5}{7}(\tfrac{1}{2}T)^{\frac{1}{2}}), \tag{4.8a}$$

$$V_\theta = (-\Omega^2 r/\beta)\,\tfrac{3}{5}(\tfrac{1}{2}T)^{\frac{1}{2}}. \tag{4.8b}$$

a (Å)	Ω (r.p.m.)	θ	$T^{\frac{1}{2}}$	$(M'-M)/M$
5×10^4	5000	3·9°	0·16	0·081
2×10^4	30000	3·9°	0·16	0·081
2×10^4	60000	5·3°	0·22	0·111
5000	5000	0·4°	0·016	0·008
5000	30000	1·0°	0·040	0·020
5000	60000	1.4°	0·056	0·028
20	5000	—	$6·5\times10^{-5}$	$3·3\times10^{-5}$
20	30000	—	$1·6\times10^{-4}$	$8·1\times10^{-5}$
20	60000	—	$2·2\times10^{-4}$	$1·1\times10^{-4}$

TABLE 2. Corrections in the 'molecular weight' of spherical particles due to the Coriolis modification of the Stokes drag law. $\nu = 0·01$ cm²/s

The 'classical' result corresponds to $T = 0$. The sphere will sediment to $O(T^{\frac{1}{2}})$ at an angle θ given by

$$\theta = \tan^{-1}(V_\theta/V_r) = -\tfrac{3}{5}(\tfrac{1}{2}T)^{\frac{1}{2}} \tag{4.9a}$$

at a speed V given by

$$V = (V_r^2+V_\theta^2)^{\frac{1}{2}} = (\Omega^2 r/\beta)(1-\tfrac{5}{7}(\tfrac{1}{2}T)^{\frac{1}{2}}). \tag{4.9b}$$

One means of finding the molecular weight M of a sample in an ultracentrifuge is to use the relation (Bowen 1970)

$$\frac{M}{N}\left(1-\frac{\rho}{\bar{\rho}}\right) = fs, \tag{4.10}$$

where N is Avogadro's number and f and s are the friction sedimentation coefficients, defined as

$$f = \frac{(\mathbf{F}_d)_r}{dr/dt}, \quad s = \frac{dr/dt}{\Omega^2 r}.$$

Equation (4.10) is usually solved for M using the Stokes drag coefficient f. To see what change the altered drag coefficient will make in molecular-weight calculations, compute M' and f', the altered molecular weight and friction coefficient in the radial direction:

$$M'/M = f'/f = 1+\tfrac{5}{7}(\tfrac{1}{2}T)^{\frac{1}{2}}. \tag{4.11}$$

Thus the percentage change in molecular weight will depend directly on the change in the radial friction coefficient. The angle through which a particle is 'deflected' is a direct measure of its size. Table 2 gives possible values. Thus it is seen that only for the smallest of particles treated should the Stokes law remain uncorrected.

The motion of the sphere gives rise to a far-field effect that has intrinsic fluid dynamical interest. In the case of rotating fluids of zero or very small viscosity, Taylor (1923) showed that a whole cylinder of fluid moves with a translating body. In contrast to the presence of a Taylor column, the present situation of motion in a fluid having significant viscosity gives rise to cubical cones (both above and below the sphere) which have angular dependence and decay along the rotation axis. Both Childress (1964) and Bretherton (1967) encountered structures involving the same similarity parameter $|z'|^{-\frac{1}{3}}(x'^2+y'^2)^{\frac{1}{2}}$, where z' is parallel to the rotation axis. An axisymmetric cone of this type was encountered by Childress (1964) in his analysis of the motion of a sphere along the rotation axis. Bretherton (1967) has examined an initial-value problem of a two-dimensional cylinder impulsively accelerated along the axis of a rotating fluid. He found a two-dimensional analogue of the cone and interpreted its structure in a way that is applicable here. The structures are composed of inertial waves that propagate along the rotation axis, have finite non-zero wavelength and zero frequency and decay through the action of viscosity. The present analysis concentrates on the Coriolis corrections to Stokes flow.

The effect of the presence of bounding surfaces was examined in §3. Two parallel planes normal to the rotation axis and rotating with the centrifuge modify the motion of a particle on the midplane. It is found that the deflexion angle θ is diminished to an extent which varies monotonically with wall separation (at least to $O(T^{\frac{1}{2}})$). The component of drag transverse to $\mathbf{F}$ is found to decrease monotonically with wall separation but the component along $\mathbf{F}$ decreases with the separation to a minimum value, after which it increases to the value given by the unbounded problem. This minimum occurs because the wall effect decreases monotonically while the Coriolis effect increases monotonically with wall separation. As a by-product of the analysis, the wall effect on a particle in a non-rotating fluid was obtained. These results agree with those of Faxén (1922), who invoked the method of images. A more important by-product is the method of analysis here. By posing the governing equations for w and ζ in matrix form, a self-adjoint matrix operator emerges. The Green's matrix function that inverts this operator is then symmetric, so that there are great simplifications in the calculations that occur. The simplifications are especially apparent when the geometry allows two of the variables to be removed by Fourier transforms or series. It seems clear that a whole class of homogeneous rotating fluid problems could be profitably handled in this way.

The support of the National Science Foundation Grants GK 31794 (Engineering Mechanics Program) and GA-353-90X (Atmospheric Sciences Program) is gratefully acknowledged. The calculations were performed at the Johns Hopkins Computation Center.

Appendix A. Multipole expansion about the sphere

The linearity of the problem implies that the effect of the sphere may be replaced by a distributed body force $\mathbf{X}(\mathbf{r})$ defined on $0 \leqslant r \leqslant 1$. The inner equations are as follows:

$$\nabla^2 v_i - T\epsilon_{ijk}\delta_{j3}v_k - \partial p/\partial x_i = X_i \qquad \text{(A 1}\,a)$$

and

$$\partial v_l/\partial x_l = 0. \qquad \text{(A 1}\,b)$$

It is well known that Green's matrix $\{\mathscr{G}_{ji}\}$ for the Stokes-flow equations ($T = 0$) may be explicitly constructed (Happel & Brenner 1965, p. 79) and is defined by the following equations:

$$L\mathscr{G}_{ji} = \nabla^2 t_{ji} - \partial P_i/\partial x_j = -4\pi\delta_{ji}\delta(\mathbf{r}-\boldsymbol{\rho}), \tag{A 2}$$

where

$$t_{ji} = \delta_{ji}\nabla^2 R(\mathbf{r},\boldsymbol{\rho}) - \partial^2 R(\mathbf{r},\boldsymbol{\rho})/\partial x_i\,\partial x_j,$$

$$P_i = -\frac{\partial}{\partial x_i}\left(\frac{1}{R(\mathbf{r},\boldsymbol{\rho})}\right),$$

$$R^2 = (x_i-\xi_i)(x_i-\xi_i)$$

and

$$\mathbf{r}-\boldsymbol{\rho} = (x_i-\xi_i)\,\mathbf{h}_i.$$

Then, expressed as a system of integral equations, the inner equations become the following:

$$v_i(\mathbf{r}) = \frac{1}{8\pi}\int_{S(\rho)}\left[t_{ji}\left(\frac{\partial v_j}{\partial \nu} - p\nu_j\right) - v_j\left(\frac{\partial t_{ji}}{\partial \nu}p_i\nu_j\right)\right]d\Sigma(\boldsymbol{\rho})$$
$$+\frac{1}{8\pi}\int_{B(\rho)}(T\epsilon_{j3k}v_k + X_j)t_{ji}\,dV(\boldsymbol{\rho}) \tag{A 3a}$$

and

$$P(\mathbf{r}) = \frac{1}{4\pi}\int_{S(\rho)}\left[\left(\frac{\partial v_j}{\partial \nu} - P\nu_j\right)\frac{\partial}{\partial \xi_j}\left(\frac{1}{R}\right) - v_j\frac{\partial}{\partial \nu}\frac{\partial}{\partial \xi_j}\left(\frac{1}{R}\right)\right]d\Sigma(\boldsymbol{\rho})$$
$$+\frac{1}{4\pi}\int_{B(\rho)}(T\epsilon_{j3k}v_k + X_j)\frac{\partial}{\partial \xi_j}\left(\frac{1}{R}\right)dV(\boldsymbol{\rho}), \tag{A 3b}$$

where $S(\rho) = \{\boldsymbol{\rho}|\rho = A\}$, $B(\boldsymbol{\rho}) = \{\boldsymbol{\rho}|\rho < A\}$, A arbitrary.

Consider the body-force contributions to the velocity as given by (A 3a):

$$\mathscr{F}_i^{(1)} = \frac{1}{8\pi}\int_{B(\rho)}X_j t_{ji}\,dV(\boldsymbol{\rho}) = \frac{1}{8\pi}\int_{B(\rho)}X_j\left(\delta_{ji}\nabla^2 R - \frac{\partial^2 R}{\partial\xi_i\partial\xi_j}\right)dV(\boldsymbol{\rho})$$
$$= \frac{1}{4\pi}\int_{B(\rho)}\frac{X_i}{R}\,dV(\boldsymbol{\rho}) - \frac{1}{8\pi}\int_{B(\rho)}X_j\frac{\partial^2 R}{\partial\xi_i\partial\xi_j}\,dV(\boldsymbol{\rho}). \tag{A 4a}$$

The body-force contribution to the pressure is given by (A 3b):

$$\mathscr{F}_i^{(2)} = \frac{1}{4\pi}\int_{B(\rho)}X_j\frac{\partial}{\partial\xi_j}\left(\frac{1}{R}\right)dV(\boldsymbol{\rho}). \tag{A 4b}$$

In order to represent the presence of the sphere in the far field a multipole expansion of the first term in (A 4a) (Morse & Feshbach 1953, p. 1276) can be performed to obtain a Laurent series about the point at infinity:

$$\frac{1}{4\pi}\int_{B(\rho)}\frac{X_i(\boldsymbol{\rho})}{R(\mathbf{r}-\boldsymbol{\rho})}dV(\boldsymbol{\rho}) \sim \frac{1}{4\pi}\left[\frac{F_i}{r} - F_{ij}\frac{\partial}{\partial x_j}\left(\frac{1}{r}\right)\right.$$
$$\left.+\frac{F_{ijk}}{2!}\frac{\partial^2}{\partial x_j\partial x_k}\left(\frac{1}{r}\right) + \ldots + (-1)^\alpha\frac{F_{ijkl\ldots}}{\alpha!}\frac{\partial^\alpha}{\partial x_j\partial x_k\partial x_l\ldots}\left(\frac{1}{r}\right) + \ldots\right], \tag{A 5}$$

where F_i, F_{ij}, F_{ijk}, etc. represent force, dipole, quadrupole, and higher contributions to the velocity field. With this information, the distributed force $\mathbf{X}(\mathbf{r})$ can be represented. The result of using this in (A 1 *a*) is as follows:

$$\nabla^2 v_i - T\epsilon_{ijk}\delta_{j3}v_k - \frac{\partial p}{\partial x_j} = -F_i\delta(\mathbf{r}) + F_{ij}\frac{\partial}{\partial x_j}\delta(\mathbf{r}) - \frac{F_{ijk}}{2!}\frac{\partial^2}{\partial x_j \partial x_k}\delta(\mathbf{r}) + \ldots, \quad \text{(A 6)}$$

where

$$F_i = \int_{B(\boldsymbol{\rho})} X_i \, dV(\boldsymbol{\rho}), \quad F_{ij} = \int_{B(\boldsymbol{\rho})} X_i \xi_j \, dV(\boldsymbol{\rho}), \quad \text{(A 7 } a, b)$$

$$F_{ijk} = \int_{B(\boldsymbol{\rho})} X_i \xi_j \xi_k \, dV(\boldsymbol{\rho}). \quad \text{(A 7 } c)$$

These integrals may be specified over B ($\rho \leqslant 1$) since $X_i \equiv 0$ for $\rho > 1$. F_i is the drag on the sphere and $F_i = -\delta_{i1}$ for a unit Stokeslet. The above scheme shows exactly how the higher moments may be computed and is therefore useful in obtaining an outer expansion of any number of terms [cf. Saffman 1965; though the factors $1/\alpha!$ are missing from his equation (3.1)]. Saffman has shown that the terms with odd numbers of indices relate to drag, lift and their moments, while even numbers relate to torque and its higher moments. The equations can be expressed in outer variables. Let $\mathbf{r}' = T^{\frac{1}{2}}\mathbf{r}$, $P' = T^{-\frac{1}{2}}P$ and $\mathbf{v}_1' = \mathbf{v}_i$ in (A 6). Since

$$\delta(\mathbf{r}'T^{-\frac{1}{2}}) = T^{\frac{3}{2}}\delta(\mathbf{r}'),$$

the result is

$$\nabla'^2 v_1' - \epsilon_{i3k}v_k' - \frac{\partial P'}{\partial x_1'} = -T^{\frac{1}{2}}F_i\delta(\mathbf{r}') + TF_{ij}\frac{\partial}{\partial x_j'}\delta(\mathbf{r}') - \ldots, \quad \text{(A 8)}$$

and the effect in the outer region of the presence of the sphere becomes ordered in powers of $T^{\frac{1}{2}}$.

Appendix B. The differential system (3.19)

The matrix operator $\mathbf{L}$ of system (3.19) is defined on two-vectors such as

$$\boldsymbol{\Psi} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \boldsymbol{\Lambda} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

$\mathbf{L}$ is easily shown to be self-adjoint on the space of these vectors (Friedman 1956, p. 148) subject to the scalar product

$$\langle \boldsymbol{\Psi}, \boldsymbol{\Lambda} \rangle = \int_{-1}^{1} (u_1 v_1 + u_2 v_2)\, dz.$$

As a result (Friedman 1956, p. 173), Green's matrix $\mathscr{G}(z,\xi)$ for $\mathbf{L}$ satisfies the symmetry condition

$$G_{ij}(z,\xi) = G_{ji}(\xi, z) \quad (i,j = 1,2).$$

Green's matrix $\mathscr{G}$ for system (3.19) can be written in the form

$$\mathscr{G} = \begin{cases} \mathbf{U}(z)\,\mathbf{A}(\xi), & z < \xi, \\ \mathbf{V}(z)\,\mathbf{B}(\xi), & z > \xi, \end{cases}$$

where
$$\mathbf{U} = \begin{pmatrix} u_1(z) & u_2(z) & u_3(z) \\ \chi_1(z) & \chi_2(z) & \chi_3(z) \end{pmatrix},$$
$$\mathbf{V} = \begin{pmatrix} v_1(z) & v_2(z) & v_3(z) \\ \eta_1(z) & \eta_2(z) & \eta_3(z) \end{pmatrix}.$$

U and **V** are solutions of the homogeneous equation and respectively satisfy the boundary conditions at $z = -1$ and $z = 1$. The 3×2 matrices **A** and **B** are unknown at the moment. If one applies the jump conditions

$$\begin{pmatrix} [G_{11,zzz}]_{\xi-}^{\xi+} & [G_{12,z}]_{\xi-}^{\xi+} \\ [G_{21,z}]_{\xi-}^{\xi+} & [G_{22,z}]_{\xi-}^{\xi+} \end{pmatrix} = \mathbf{I},$$

then one obtains the linear algebraic system

$$\mathbf{W\Gamma} = \mathbf{E} \quad \text{or} \quad \mathbf{\Gamma} = \mathbf{W}^{-1}\mathbf{E},$$

where

$$\mathbf{W} = \begin{pmatrix} v_1 & v_2 & v_3 & u_1 & u_2 & u_3 \\ v_1' & v_2' & v_3' & u_1' & u_2' & u_3' \\ v_1'' & v_2'' & v_3'' & u_1'' & u_2'' & u_3'' \\ v_1''' & v_2''' & v_3''' & u_1''' & u_2''' & u_3''' \\ \eta_1 & \eta_2 & \eta_3 & \chi_1 & \chi_2 & \chi_3 \\ \eta_1' & \eta_2' & \eta_3' & \chi_1' & \chi_2' & \chi_3' \end{pmatrix}$$

is the Wronksian matrix of the fundamental matrix $\mathbf{\Phi}$,

$$\mathbf{\Gamma} = \begin{pmatrix} \mathbf{B} \\ -\mathbf{A} \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lagrange's identity (see Ince 1926, p. 124) gives that there exists a constant matrix $\mathscr{P}$ such that
$$\mathscr{P} = \tilde{\mathbf{W}}\mathscr{R}\mathbf{W},$$

where $\mathscr{R}$ depends on the coefficients in **L**. Here

$$\mathscr{R} = \begin{pmatrix} 0 & -2\lambda^2 & 0 & 1 & -T_L & 0 \\ 2\lambda^2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ T_L & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Hence $\mathbf{\Gamma} = \mathscr{P}^{-1}\mathbf{\Phi}$. Since the system (3.19) is self-adjoint,

$$\mathscr{P} = \begin{pmatrix} \mathbf{O} & \mathbf{P} \\ -\mathbf{P} & \mathbf{O} \end{pmatrix},$$

where $\mathbf{O}$ is the 3×3 zero matrix and $\mathbf{P}$ is symmetric. As a result, the Green's matrix has the form

$$\mathscr{G}(z,\xi) = \begin{cases} -\mathbf{V}(\xi)\,\mathbf{P}^{-1}\tilde{\mathbf{U}}(\xi), & z > \xi, \\ -\mathbf{U}(z)\,\mathbf{P}^{-1}\tilde{\mathbf{V}}(\xi), & z < \xi. \end{cases}$$

The matrices $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{P}$ are obtained from the following:

$$v_1(z) = \sum_{n=1}^{3} \mu_n \sinh\gamma_n(1-z),$$

$$v_2(z) = \cosh\gamma_3(1-z) - \cosh\gamma_1(1-z),$$

$$v_3(z) = \cosh\gamma_1(1-z) - \cosh\gamma_2(1-z),$$

$$u_i(z) = v_i(-z), \quad i = 1,2,3,$$

$$\eta_1(z) = T_L \sum_{n=1}^{3} \frac{\mu_n\gamma_n\cosh\gamma_n(1-z)}{\gamma_n^2-\lambda^2},$$

$$\eta_2(z) = T_L\left[\frac{\gamma_3\sinh\gamma_3(1-z)}{\gamma_3^2-\lambda^2} - \frac{\gamma_1\sinh\gamma_1(1-z)}{\gamma_1^2-\lambda^2}\right],$$

$$\eta_3(z) = T_L\left[\frac{\gamma_1\sinh\gamma_1(1-z)}{\gamma_1^2-\lambda^2} - \frac{\gamma_2\sinh\gamma_2(1-z)}{\gamma_2^2-\lambda^2}\right],$$

$$\chi_i(z) = -\eta_i(-z), \quad i = 1,2,3.$$

The constants γ_i and μ_i are determined by the three equations

$$(\gamma_i^2-\lambda^2)^3 + T_L^2\gamma_i^2 = 0, \quad \sum_{n=1}^{3}\mu_n\gamma_n = 0, \quad \sum_{n=1}^{3}\frac{\gamma_n\mu_n}{\gamma_n^2-\lambda^2} = 0.$$

The latter two follow from the boundary conditions. Choose

$$\mu_1 = (\gamma_1^2-\lambda^2)(\gamma_2^2-\gamma_3^2)/\gamma_1,$$

$$\mu_2 = (\gamma_2^2-\lambda^2)(\gamma_3^2-\gamma_1^2)/\gamma_2,$$

$$\mu_3 = (\gamma_3^2-\lambda^2)(\gamma_1^2-\gamma_2^2)/\gamma_3.$$

The matrix $\mathbf{P}$ then has the form

$$P_{11} = C\sum_{n=1}^{3}\mu_n\sinh 2\gamma_n, \quad P_{21} = C(\cosh 2\gamma_3 - \cosh 2\gamma_1),$$

$$P_{31} = C(\cosh 2\gamma_1 - \cosh 2\gamma_2),$$

$$P_{12} = P_{21}, \quad P_{22} = C\left(\frac{\sinh 2\gamma_3}{\mu_3} + \frac{\sinh 2\gamma_1}{\mu_1}\right), \quad P_{32} = -C\frac{\sinh 2\gamma_1}{\mu_1},$$

$$P_{13} = P_{31}, \quad P_{23} = P_{32}, \quad P_{33} = C\left(\frac{\sinh 2\gamma_1}{\mu_1} + \frac{\sinh 2\gamma_2}{\mu_2}\right),$$

where $C = \sum_{n=1}^{3}\mu_n\gamma_n^3$.

REFERENCES

BERMAN, A. S. 1966 *J. Nat. Cancer Inst. Monograph*, **21**, 41.
BLAKE, J. R. 1971 *Proc. Camb. Phil. Soc.* **70**, 303.
BOWEN, T. J. 1970 *An Introduction to Ultracentrifugation*. Interscience.
BRETHERTON, F. P. 1967 *J. Fluid Mech.* **28**, 545.
CHENG, P. Y. & SCHACHMAN, H. K. 1955 *J. Polymer Sci.* **16**, 19.
CHILDRESS, S. 1964 *J. Fluid Mech.* **20**, 305.
COX, R. G. & BRENNER, H. 1967 *J. Fluid Mech.* **28**, 391.
FAXÉN, H. 1922 *Ann. Phys.* **68**, 89.
FRIEDMAN, B. 1956 *Principles and Techniques of Applied Mathematics*. Wiley.
FUJITA, H. 1962 *The Mathematical Theory of Sedimentation Analysis*. Academic.
GEL'FAND, I. M. & SHILOV, G. E. 1964 *Generalized Functions*, vol. 1. Academic.
GREENSPAN, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.
HAPPEL, J. & BRENNER, H. 1965 *Low Reynolds Number Hydrodynamics*. Prentice-Hall.
HOOYMAN, G. J., HOLTAN, H., MAZUR, P. & DE GROOT, S. R. 1953 *Physica*, **19**, 1095.
INCE, E. L. 1926 *Ordinary Differential Equations*. Longmans.
KAPLUN, S. & LAGERSTROM, P. A. 1957 *J. Math. Mech.* **6**, 585.
LANGFORD, D. 1968 *J. Appl. Mech.* **35**, 683.
MOORE, D. W. & SAFFMAN, P. G. 1969 *J. Fluid Mech.* **39**, 831.
MORSE, P. M. & FESHBACH, H. 1953 *Methods of Theoretical Physics*. McGraw-Hill.
SAFFMAN, P. G. 1965 *J. Fluid Mech.* **22**, 385.
SCHACHMAN, H. K. 1959 *Ultracentrifugation in Biochemistry*. Academic.
SHARP, D. G. & BEARD, J. W. 1950 *J. Biol. Chem.* **185**, 247.
STEWARTSON, K. 1957 *J. Fluid Mech.* **3**, 17.
TAYLOR, G. I. 1923 *Proc. Roy. Soc.* A **104**, 213.